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Random Walk in Dynamically Disordered Chains: Poisson White Noise Disorder

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Exact solutions are given for a variety of models of random walks in a chain with time-dependent disorder. Dynamic disorder is modeled by white Poisson noise. Models with site-independent (global) and site-dependent (local) disorder are considered. Results are described in terms of an affective random walk in a nondisordered medium. In the cases of global disorder the effective random walk contains multistep transitions, so that the continuous limit is not a diffusion process. In the cases of local disorder the effective process is equivalent to usual random walk in the absence of disorder but with slower diffusion. Difficulties associated with the continuous-limit representation of random walk in a disordered chain are discussed. In particular, we consider explicit cases in which taking the continuous limit and averaging over disorder sources do not commute.

KEY WORDS: Random Walk; Master Equation; Poisson Noise; Disordered Systems; Diffusion.

1. INTRODUCTION

A standard model to study transport properties in the general context of disordered systems is random walk in a disordered lattice.[1,2] In this framework, several models have been studied in which the transition probability per unit time to jump from one site to a neighboring one is a time-independent random variable. We refer to these models as static disorder models. More recently the problem has been posed of random walk in media with dynamic disorder.[3-6] In these problems the transition probability per unit time which appears in the master equation (ME)

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describing ordinary random walk is replaced by a time-dependent random process. Random walk in dynamically disordered systems is of relevance for diffusion in microemulsion globules[3] and ionic conduction in polymeric solid electrolytes, among other systems.[5]

Not many exact results seem to exist for the problem of random walk in systems with dynamic disorder. An early failed attempt in this direction appears in ref. 7, where the problem is addressed considering Gaussian noise disorder and taking a continuous limit of the random walk ME. We come back here to these problems, our aim in this paper being to explore a variety of models with dynamic disorder for which explicit exact results are given. In our models, dynamic disorder is introduced by transition probabilities which fluctuate as Poisson white noise. The use of white noise partly destroys correlations among different spatial points. This simplification permits an explicit discussion of several aspects which also appear in models with nonwhite noise. Our results should offer a useful guideline in these more complicated cases. The use of Poisson noise permits us to maintain the positivity of the transition probabilities so that well-defined stochastic models appear at each step of the calculation. The key idea of our development, which is also applicable to other situations with nonwhite dynamic disorder,[8] is the exact derivation of effective master equations (EME): Starting from an ordinary ME describing random walk in an ordered medium, the introduction of stochastic transition probabilities defines a stochastic master equation (SME). The average of the SME over the noise sources modeling dynamic disorder leads to an EME. The EME defines a generalized random walk in an effective nondisordered medium. In our models this generalized random walk is still a Markov process defined by an ME with effective transition rates. In addition, the continuous limit of the EME is always well defined, while other methods to obtain continuous representations of the problem are cumbersome, often lead to mistakes, and in the best situation require careful interpretation of ill-defined quantities.

Two different classes of models are introduced in Section 2. In a first class (global disorder) the noise sources are site independent. In the second class (local disorder) the noise sources are site dependent. This second class contains the dynamic generalization of the random barrier (RB) and random trap (RT) models.[1,2] First, in Section 2 we recall basic facts about random walk in an ordered chain. In particular, we give an interpretation in terms of sample paths of the nondiffusive quasicontinuous approximation recently introduced by Doering *et al.*[9] Section 3 includes our central results. We find that starting with an ME which only includes nearest-neighbor transition rates, the EME description contains multistep transitions per unit time in the case of global disorder, but only modified nearest-neighbor transitions in the case of local disorder. In the case of global disorder we find faster diffusion or no modification of the diffusion constant, depending on the symmetry properties of the model. When the disorder is of local nature it always leads to a slower diffusion. Our formulation leads to diffusion equations for the continuous limit of the models with local disorder that we consider. However, the continuous limits of the models with global disorder are not diffusion equations, so that the underlying sample paths are not continuous, but contain jump processes. In a sense, white dynamic disorder seems to have more important consequences for global than local disorder. All these results are interpreted in terms of sample paths. Finally, we show in Section 4 that taking a continuous limit representation does not generally commute with the averaging over noise associated with disorder. We make clear the convenience of taking the average that leads to an EME before a continuous limit is considered.

2. RANDOM WALK MODELS

2.1. Nondisordered Chain

Ordinary random walk in a chain is defined by the ME for the probability $P(N, t)$ of finding the random walker at site N at time t:

$$\partial_t P(N, t) = \mu(E^+ + E^- - 2)\, P(N, t) \tag{2.1}$$

where $E^\pm \equiv e^{\pm \partial/\partial_N}$ are shifting operators:

$$E^\pm f(N) = f(N \pm 1) \tag{2.2}$$

We are considering a symmetric random walk with jumps only between nearest-neighbor sites at a rate μ. The rate μ fixes the time scale of the process. Introducing a lattice spacing l and a variable $x = Nl$, the probability density $P^l(x, t) \equiv (1/l)\, P(N = x/l, t)$ obeys the equation

$$\partial_t P^l(x, t) = \mu(e^{+l\partial/\partial x} + e^{-l\partial/\partial x} - 2)\, P^l(x, t) \tag{2.3}$$

In the continuous limit $\mu \to \infty$, $l \to 0$, with $\mu l^2 = D$, (2.3) becomes the diffusion equation for a diffusion process $x(t)$:

$$\partial_t P(x, t) = D \frac{\partial^2}{\partial x^2} P(x, t) \tag{2.4}$$

An interesting alternative representation of (2.3) is given by a stochastic differential equation (SDE), driven by Poisson white noise sources, for the stochastic process $x(t)$. Poisson white noise is defined as a limit in which the duration of the pulses of a generalized Poisson process $z(t)$ (shot noise) goes to zero.[10–12] The process $z(t)$ is defined by

$$z(t) = \sum_{i=1}^{n(t)} \omega_i h(t - t_i) \tag{2.5a}$$

where $n(t)$ is a Poisson counting process with probability

$$P(n(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{2.6}$$

The times t_i are uniformly distributed in the interval $(0, t)$, and occur with mean frequency λ. The function $h(t - t_i)$ is a pulse attached to the time t_i such that $h(t - t_i) = 0$ for $t < t_i$. The pulses are weighted by ω_i, which are random independent variables with a probability distribution $\rho(\omega)$ and mean value $\bar{\omega}$. Examples of realizations of generalized Poisson noise with rectangular and exponential pulses are shown in ref. 11. In the limit in which the pulses $h(t - t_i)$ become delta functions, the process $z(t)$ becomes a white Poisson noise. White Poisson noise $\eta(t)$ is then given by a sequence of delta peaks at random times t_i determined by a Poisson counting process. Imposing a zero mean value, $\eta(t)$ is defined as

$$\eta(t) = \sum_{i=1}^{n(t)} \omega_i \delta(t - t_i) - \lambda \bar{\omega} \tag{2.5b}$$

Using the definition (2.5b), Eq. (2.3) is equivalent to

$$\dot{x}(t) = \eta^+(t) - \eta^-(t) \tag{2.7}$$

where $\eta^{\pm}(t)$ are two independent white Poisson noises with the same parameter $\lambda = \mu$ and same fixed pulse amplitude $\omega_i = l$. The dot denotes the time derivative. Equation (2.3) can be obtained from (2.7) defining $P(x, t) \equiv \overline{\delta(x - x(t))}$, where the bar denotes average over the noise, and using an important formula for the average of $\eta(t)$ with a functional of $\eta(t)$,[12]

$$\overline{\eta(t)\, \phi[\eta]} = \lambda \sum_{n=2}^{\infty} \frac{\{\omega^n\}}{n!} \overline{\frac{\delta^{n-1} \phi[\eta]}{\delta \eta(t)^{n-1}}} \tag{2.8}$$

where $\{\omega^n\}$ are averages taken with $\rho(\omega)$. One finds[12]

$$\partial_t P(x, t) = \lambda \left[\sum_{k=0}^{\infty} \frac{\{\omega^k\}}{k!} \partial_x^k + \sum_{k=0}^{\infty} \frac{\{\omega^k\}}{k!} (-\partial_x)^k - 2 \right] P(x, t) \tag{2.9}$$

which reproduces (2.3) when $\rho(\omega)=\delta(\omega-l)$, $\lambda=\mu$. The diffusion process (2.4) for $x(t)$ is obtained in the continuous limit of the representation (2.7) noting that for $\mu\to\infty$, $l\to 0$ with $\mu l^2=D$, $\eta^+(t)-\eta^-(t)$ becomes a Gaussian white noise of zero mean value and correlation

$$\langle[\eta^+(t_1)-\eta^-(t_1)][\eta^+(t_2)-\eta^-(t_2)]\rangle\to 2D\delta(t_1-t_2) \qquad (2.10)$$

Also in connection with the representation (2.7) of a free random walk it is possible to give an interpretation of the quasicontinuous approximation (QCA) to (2.1) recently introduced by Doering *et al.*[9] Indeed, if we now take for the pulse amplitude an exponential distribution with mean value $\sqrt{\varepsilon}$, $\rho(\omega)=(1/\sqrt{\varepsilon})\exp(-\omega/\sqrt{\varepsilon})$, we obtain from (2.9)

$$\partial_t P(x,t)=D\frac{\partial_x^2}{1-\varepsilon\partial_x^2}P(x,t) \qquad (2.11)$$

Doering *et al.* obtained (2.11) as an approximation to the Kramers–Moyal expansion of (2.3) which contains derivatives to all orders and which preserves positivity of $P(x,t)$.[3] Equation (2.11) goes beyond the diffusion limit, incorporating discrete lattice effects ($l\neq 0$). In this sense it was introduced as a correction to the diffusion equation (2.4). We have here obtained (2.11) by allowing jumps with an exponential distribution of the jump size in a discrete random walk. This derivation answers the question raised[9] of the possible interpretation of the QCA (2.11) in terms of sample paths: The QCA is equivalent to a Markovian random walk with exponentially distributed jump size.

2.2. Models with Dynamic Disorder

As a first set of models for random walk in a dynamically disordered system we consider situations in which the jump rate μ becomes a stochastic function of time which is independent of the lattice site. This stochasticity of μ might model random changes in the energy of the random walker, for instance, through an applied external field.[13] When the random part of μ is site dependent (local disorder), one is rather modeling random changes in the local potential of the medium in which the particle is hopping. Within the case of global disorder and in order to explore the consequences of symmetry properties, we consider two models. In the global-symmetric model (GS), the jump rate μ in (2.1) is substituted by a random process. As a consequence, the transition probability per unit

[3] For a recent discussion of the positivity of solutions of truncated Kramers–Moyal expansions see Risken and Vollmer.[18]

time is the same for jumps forward and backward. Replacing in (2.1) μ by $\mu+\alpha\xi(t)$, where $\xi(t)$ is a random process, the GS model is defined by

$$\text{GS} \qquad \partial_t P(N,t)=\mu(E^+ + E^- - 2)\,P(N,t)+\alpha\xi(t)(E^+ + E^- - 2)\,P(N,t) \tag{2.12}$$

α in (2.12) is a scaling parameter. Equations like (2.12) will be termed stochastic master equations (SME) in the sense that the transition probabilities are themselves random functions of time.

A global asymmetric model (GA) can be constructed by introducing two independent random processes $\xi^{\pm}(t)$ for the jump rates forward and backward. In this case the transition rate per unit time for jumps $N\to N+1$ is different than for jumps $N\to N-1$, although both rates are independent of N. The SME defining this model is

$$\begin{aligned}\text{GA} \qquad \partial_t P(N,t)=&\,\mu(E^+ + E^- - 2)\,P(N,t)+\alpha\xi^-(t)(E^+ - 1)\,P(N,t)\\ &+\alpha\xi^+(t)(E^- - 1)\,P(N,t)\end{aligned} \tag{2.13}$$

Equations (2.12) and (2.13) define $P(N,t)$ as a functional of the noise $\xi(t)$ or $\xi^{\pm}(t)$. For each realization of the noise these equations still have the meaning of an ME provided that the stochastic transition rates $\mu+\alpha\xi(t)$ or $\mu+\alpha\xi^{\pm}(t)$ remain positive. This requirement places severe limitations on the possible choices of the noise. In particular, Gaussian noise is not allowed, because it has unbounded positive and negative realizations. Following past experience with a similar problem,[12] we choose here to model the random processes $\xi(t)$ and $\xi^{\pm}(t)$ by Poisson white noise as defined in (2.5b) with parameter λ and mean value $\{\omega\}\equiv\bar{\omega}$. We note that in (2.13) $\xi^+(t)$ and $\xi^-(t)$ are taken as independent processes, although with the same parameters λ and $\bar{\omega}$. A Poisson white noise is bounded from below:

$$\xi(t)\geqslant -\lambda\bar{\omega}, \qquad \xi^{\pm}(t)\geqslant -\lambda\bar{\omega} \tag{2.14}$$

so that the positivity requirement for the transition rates is satisfied whenever $\mu-\alpha\lambda\bar{\omega}\geqslant 0$. Fulfilling this condition and with our choice of $\xi(t)$ and $\xi^{\pm}(t)$, (2.12)–(2.13) characterize well-defined stochastic processes which model random walks in systems with global dynamic disorder.

We now introduce models with local disorder. We first consider the generalization to the dynamic case of two well-known models of static disorder, namely the random trap (RT) and random barrier (RB) models.[1,2] Starting from a general one-step ME,

$$\begin{aligned}\partial_t P(N,t)=&\,W^+(N-1,t)\,P(N-1,t)+W^-(N+1,t)\,P(N+1,t)\\ &-[W^+(N,t)+W^-(N,t)]\,P(N,t)\end{aligned} \tag{2.15}$$

the symmetry properties of the two models are such that in the RT case the transition rate from a site N is the same for transitions forward and backward, while in the RB case the transition rate $N \to N+1$ is the same as for the transition $N+1 \to N$.

In order to model a system with local dynamic disorder, one associates an independent Poisson white noise $\xi_N(t)$ with each site N. We take the same parameter values λ and $\bar{\omega}$ for any $\xi_N(t)$. For the generalization of the RT model to dynamic disorder we take

$$W^-(N, t) = W^+(N, t) = \mu + \alpha \xi_N(t) \tag{2.16}$$

so that the appropiate SME reads

$$\text{RT} \qquad \partial_t P(N, t) = \mu(E^+ + E^- - 2)\, P(N, t) + (E^+ - E^- - 2)\, \alpha \xi_N(t)\, P(N, t) \tag{2.17}$$

For the generalization of the RB model we take

$$W^-(N, t) = W^+(N-1, t) = \mu + \alpha \xi_N(t) \tag{2.18}$$

$$W^-(N+1, t) = W^+(N, t) = \mu + \alpha \xi_{N+1}(t) \tag{2.19}$$

and the appropriate SME reads

$$\begin{aligned} \text{RB} \qquad \partial_t P(N, t) = {} & \mu(E^+ + E^- - 2)\, P(N, t) + \alpha \xi_{N+1}(t)(E^+ - 1)\, P(N, t) \\ & + \alpha \xi_N(t)(E^- - 1)\, P(N, t) \end{aligned} \tag{2.20}$$

In the limit in which $\xi_N(t)$ becomes site independent both models (2.17) and (2.20) reproduce the GS model (2.12). A model with local disorder and intrinsic asymmetry in the transition rates can be introduced by choosing

$$W^\pm(N, t) = \mu + \alpha \xi_N^\pm(t) \tag{2.21}$$

so that the associated SME is

$$\begin{aligned} \partial_t P(N, t) = {} & \mu(E^+ + E^- - 2)\, P(N, t) + (E^+ - 1)\, \alpha \xi_N^-(t)\, P(N, t) \\ & + (E^- - 1)\, \alpha \xi_N^+(t)\, P(N, t) \end{aligned} \tag{2.22}$$

For site-independent noise, (2.22) gives rise to the GA model (2.13).

The three SMEs just introduced for systems with local disorder define well-behaved ME when the same condition ($\mu - \alpha\lambda\bar{\omega} \geq 0$) as for global disorder is satisfied. In the limit in which the random processes $\xi_N(t)$ become time-independent random variables one recovers the well-known models of static disorder. This would correspond to considering nonwhite noise $\xi_N(t)$ with a finite correlation time τ and taking the limit $\tau \to \infty$.

3. EFFECTIVE MASTER EQUATIONS

In this section we derive EMEs for the models introduced above. The basic idea is to consider that an SME defines $P(N, t)$ as a functional of the noise sources, so that, averaging over the realizations of the noise, one arrives at a new ME, the EME, for the averaged probability distribution $\overline{P(N, t)}$. The EME for $\overline{P(N, t)}$ defines a Markov process which includes the effects of the disorder in the random walk properties. The fact that the EME can be obtained exactly and that it defines a Markov process is a consequence of the use of white noise, which greatly simplifies the problem. Statistical properties are easily obtained from the EME.

3.1. Gobal Disorder

We indicate by $\overline{(...)}$ the average over the realizations of $\xi(t)$. Let us first consider the average of the GS model (2.12). The only difficulty appears in the calculation of $\overline{\xi(t)\, P(N, t)}$. This average can be done using the general formula (2.8):

$$\overline{\xi(t)\, P(N, t)} = \lambda \sum_{n=2}^{\infty} \frac{\{\omega^n\}}{n!} \frac{\overline{\delta^{n-1} P(N, t)}}{\delta \xi(t)^{n-1}}$$

$$= \lambda \sum_{n=2}^{\infty} \frac{\{\omega^n\}}{n!} [\alpha(E^+ + E^- - 2)]^{n-1}\, \overline{P(N, t)} \tag{3.1}$$

Substituting (3.1) in the average of (2.12), we arrive at the EME for the GS model:

$$\partial_t \overline{P(N, t)} = [(\mu - \lambda\alpha\bar{\omega})(E^+ + E^- - 2) + \lambda\{e^{\alpha\omega(E^+ + E^- - 2)} - 1\}]\, \overline{P(N, t)} \tag{3.2}$$

where $\{...\}$ is the average over the distribution $\rho(\omega)$ for the pulse amplitudes. For fixed amplitudes $\rho(\omega) = \delta(\omega - \bar{\omega})$, the average is directly read from (3.2). For an exponential distribution $\rho(\omega) = (1/\bar{\omega}) \exp(-\omega/\bar{\omega})$ we find

$$\partial_t \overline{P(N, t)} = \left[\mu(E^+ + E^- - 2) + \lambda\alpha^2\bar{\omega}^2 \frac{(E^+ + E^- - 2)^2}{1 - \alpha\bar{\omega}(E^+ + E^- - 2)} \right] \overline{P(N, t)} \tag{3.3}$$

The EME for this model can be easily solved by Fourier transformation:

$$\hat{P}_q(t) = \sum_N e^{-iqN}\, \overline{P(N, t)} \tag{3.4}$$

The solution with initial condition $\overline{P(N, 0)} = \delta_{N,0}$ is

$$\hat{P}_q(t) = e^{-\varepsilon(q)t} \tag{3.5}$$

where

$$\varepsilon_{\text{F}}(q) = 2(\mu - \lambda\alpha\bar{\omega})(1 - \cos q) + \lambda[1 - \exp[-2\alpha\bar{\omega}(1 - \cos q)]] \tag{3.6}$$

for fixed (F) amplitude of the noise pulses and

$$\varepsilon_{\text{E}}(q) = 2\mu(1 - \cos q) - \lambda\alpha^2\bar{\omega}^2 \frac{[2(\cos q - 1)]^2}{1 + 2\alpha\bar{\omega}(1 - \cos q)} \tag{3.7}$$

for exponentially (E) distributed amplitudes. The statistical moments of $\overline{P(N, t)}$ are obtained from the usual formula

$$\langle \overline{N^m} \rangle = (-i)^m \, \partial_q^m \hat{P}_q(t)|_{q=0}$$

The notation $\langle \ldots \rangle$ indicates the average over $P(N, t)$ and $\langle \overline{\ldots} \rangle$ indicates the average over $\overline{P(N, t)}$ obtained by averaging $\langle \ldots \rangle$ over the realizations of $\xi(t)$. Results for the second and fourth moments are given in Table I. These results indicate that symmetric global white dynamic disorder does not change the diffusional behavior of $\langle \overline{N^2} \rangle$, but it does change the short-time behavior of $\langle \overline{N^4} \rangle$.

The EME for the GA model (2.13) is obtained following the same method. From (2.8) we now have

$$\overline{\xi^{\pm}(t) \, P(N, t)} = \frac{\lambda}{\alpha(E^{\mp} - 1)} \{ e^{\alpha\omega(E^{\mp} - 1)} - \alpha\omega(E^{\mp} - 1) - 1 \} \, \overline{P(N, t)} \tag{3.8}$$

so that

$$\begin{aligned} \partial_t \overline{P(N, t)} = {} & [(\mu - \lambda\alpha\bar{\omega})(E^+ + E^- - 2) \\ & + \lambda\{ e^{\alpha\bar{\omega}(E^+ - 1)} + e^{\alpha\bar{\omega}(E^- - 1)} - 2 \}] \, \overline{P(N, t)} \end{aligned} \tag{3.9}$$

The explicit form of the EME for exponentially distributed amplitudes becomes

$$\begin{aligned} \partial_t \overline{P(N, t)} = {} & \mu(E^+ + E^- - 2) \, \overline{P(N, t)} + \lambda\alpha^2\bar{\omega}^2 \\ & \times \frac{(1 + \alpha\bar{\omega})(E^{+2} + E^{-2} - 2) - 2(1 + 2\alpha\bar{\omega})(E^+ + E^- - 2)}{1 - \alpha\bar{\omega}(1 + \alpha\bar{\omega})(E^+ + E^- - 2)} \, \overline{P(N, t)} \end{aligned} \tag{3.10}$$

Table I

	Global disorder		Local disorder	
	GS model	GA model	RB or RT model	Asymmetric model
$\langle N^2\rangle_F$	$2\mu t$	$2(\mu+\lambda\alpha^2\bar{\omega}^2)\,t$	$2[\mu-\frac{1}{2}\lambda(e^{-2\alpha\bar{\omega}}-1+2\alpha\bar{\omega})]\,t$	$2[\mu-\lambda(e^{-\alpha\bar{\omega}}-1+\alpha\bar{\omega})]\,t$
$\langle N^2\rangle_E$	$2\mu t$	$2(\mu+2\lambda\alpha^2\bar{\omega}^2)\,t$	$2\left(\mu-\frac{2\lambda\bar{\omega}^2\alpha^2}{1+2\alpha\bar{\omega}}\right)t$	$2\left(\mu-\frac{\lambda\alpha^2\bar{\omega}^2}{1+\alpha\bar{\omega}}\right)t$
$\langle N^4\rangle_F$	$(2\mu+12\lambda\alpha^2\bar{\omega}^2)\,t + 12(\mu t)^2$	$[2\mu+2\lambda\alpha^2\bar{\omega}^2(7+6\alpha\bar{\omega}+\alpha^2\bar{\omega}^2)]\,t + 12(\mu+\lambda\alpha^2\bar{\omega}^2)^2\,t^2$	$2\left[\mu-\frac{\lambda}{2}(e^{-2\alpha\bar{\omega}}-1+2\alpha\bar{\omega})\right]t + 12\left[\mu-\frac{\lambda}{2}(e^{-2\alpha\bar{\omega}}-1+2\alpha\bar{\omega})\right]^2 t^2$	$2[\mu-\lambda(e^{-\alpha\bar{\omega}}-1+\alpha\bar{\omega})]\,t + 12[\mu-\lambda(e^{-\alpha\bar{\omega}}-1+\alpha\bar{\omega})]^2\,t^2$
$\langle N^4\rangle_E$	$(2\mu+24\lambda\alpha^2\bar{\omega}^2)\,t + 12(\mu t)^2$	$[2\mu+4\lambda\alpha^2\bar{\omega}^2(7+18\alpha\bar{\omega}+12\alpha^2\bar{\omega}^2)]\,t + 12(\mu+2\lambda\alpha^2\bar{\omega}^2)^2\,t^2$	$2\left(\mu-\frac{2\lambda\alpha^2\bar{\omega}^2}{1+2\alpha\bar{\omega}}\right)t + 12\left(\mu-\frac{2\lambda\alpha^2\bar{\omega}^2}{1+2\alpha\bar{\omega}}\right)^2 t^2$	$2\left(\mu-\lambda\frac{\alpha^2\bar{\omega}^2}{1+\alpha\bar{\omega}}\right)t + 12\left(\mu-\lambda\frac{\alpha^2\bar{\omega}^2}{1+\alpha\bar{\omega}}\right)^2 t^2$

The solution of the EME can also be given by (3.5) with

$$\varepsilon_{\mathrm{F}}(q)=2(\mu-\alpha\lambda\bar{\omega})(1-\cos q)-\lambda[e^{\alpha\bar{\omega}(e^{-iq}-1)}+e^{\alpha\bar{\omega}(e^{iq}-1)}-2] \tag{3.11}$$

$$\varepsilon_{\mathrm{E}}(q)=2\mu(1-\cos q)+\lambda\alpha^2\bar{\omega}^2 \times\frac{2(1+\alpha\bar{\omega})(1-\cos 2q)-4(1+2\alpha\bar{\omega})(1-\cos q)}{1+2\alpha\bar{\omega}(1+\alpha\bar{\omega})(1-\cos q)} \tag{3.12}$$

The associated first moments are also given in Table I. These results indicate that the existence of normal diffusion $\langle\overline{N^2}\rangle \propto t$ is not changed, but contrary to what happens in the symmetrical model, the value of the coefficient $d=\langle\overline{N^2}\rangle/(2t)$ is changed by the presence of disorder, giving here a faster diffusion. The behavior of the fourth moment is changed both for short and long times.

It is interesting to understand the EMEs (3.2) and (3.9) in terms of effective transition probabilities. A remarkable fact is that (3.2) and (3.9) imply the existence of nonvanishing probabilities for transitions $N\to N\pm n$ with arbitrary step n. In order to identify such effective transition probabilities per unit time $\bar{W}(N\to N\pm n)$, one has to expand the exponentials involving the shifting operators $E^{\pm}$ and rearrange the resulting series. For the GA model, (3.9) can be rewritten as

$$\partial_t\overline{P(N,t)}=\Bigg[2\lambda(\{e^{-\alpha\omega}\}-1+\alpha\bar{\omega})-2\mu+(E^{+}+E^{-})(\mu-\lambda\alpha\bar{\omega}+\lambda\alpha\{\omega e^{-\alpha\omega}\}) + \sum_{n=2}^{\infty}(E^{+n}+E^{-n})\frac{\lambda}{n!}\{\alpha^n\omega^n e^{-\alpha\omega}\}\Bigg]\overline{P(N,t)} \tag{3.13}$$

so that

$$\bar{W}(N\to N\pm 1)=\mu-\lambda\alpha\bar{\omega}+\lambda\alpha\{\omega e^{-\alpha\omega}\} \tag{3.14}$$

$$\bar{W}(N\to N\pm n)=\frac{\lambda\alpha^n}{n!}\{\omega^n e^{-\alpha\omega}\},\qquad n>1 \tag{3.15}$$

Likewise, for the GS model (3.2), it is found that

$$\bar{W}(N\pm n)=\delta_{n,1}(\mu-\lambda\alpha\bar{\omega})+\lambda\alpha^n\left\{\omega^n e^{-2\alpha\omega}\sum_{r=0}^{\infty}\frac{(\alpha\omega)^{2r}}{r!\,(n+r)!}\right\} \tag{3.16}$$

The existence of these multistep transitions can be understood in terms of the representation (2.7) of normal random walk. In the absence of disorder the random walker jumps forward or backward following the delta pulses of a given realization of $\eta^{\pm}(t)$. The mean time between pulses μ^{-1} fixes the transition probability per unit time for moving one step of size

$n=1$. Jumps of size $n>1$ have zero probability. In this picture, our models of dynamic disorder correspond to the assumption of a stochastic mean time between pulses. The substitution of μ by $\mu+\alpha\xi^{\pm}(t)$ implies that the stochastic mean time between pulses $[\mu+\alpha\xi^{\pm}(t)]^{-1}$ vanishes at times at which $\xi^{\pm}(t)$ has a delta pulse. This occurs, on the average, at time intervals λ^{-1}. When the mean time between pulses of $\eta^{\pm}(t)$ vanishes there is an accumulation of pulses, which gives rise to effective transition rates of arbitrary size $n>1$. This idea of the dynamic process permits us to give a physical argument to obtain directly (3.14)–(3.15):

$$\bar{W}(N\to N+n)=\lim_{\delta\to 0}\frac{P(N(\delta)=n)}{\delta} \tag{3.17}$$

where $P(N(\delta)=n)$ is the probability that $\eta^{+}(t)$ has n pulses in the time interval δ for a given realization of $\xi^{+}(t)$. Given the Poissonian character of $\eta^{+}(t)$,

$$P(N(\delta)=n)=\exp\left[-\int_t^{t+\delta} ds[\mu+\alpha\xi^{+}(s)]\right] \times\frac{1}{n!}\left[\int_t^{t+\delta} ds\,[\mu+\alpha\xi^{+}(s)]\right]^n \tag{3.18}$$

Since in (3.17) we only need $P(N(\delta)=n)$ to first order in δ, and, in this order, the probability of a delta pulse of $\xi^{+}(s)$ in the interval δ is $\lambda\delta$, we can calculate (3.17) for $n>1$ by replacing in (3.18) $\xi^{+}(s)$ by $\omega_i\delta(s-t_i)-\lambda\bar{\omega}$ (with $t<t_i<t+\delta$) and multiplying the resulting expression by $\lambda\delta$. This reproduces (3.15). To obtain (3.14), one needs to add the probability $P(N(\delta)=1)$ when there is no pulse of ξ.

The above argument to obtain (3.14)–(3.15) makes implicit use of the global character of the disorder and of the independence of ξ^{+} and ξ^{-}. This is seen in the dynamic path considered in (3.17)–(3.18). This path consists in n jumps forward through the n sites between N and $N+n$. The global character is taken into account because the same noise $\xi^{+}(s)$ is attached to each site, so that the probability of having a delta pulse of $\xi^{+}(s)$ in the interval δ in each intermediate site is given by the same factor $\lambda\delta$. The independence of $\xi^{+}(s)$ and $\xi^{-}(s)$ is used because paths of $n+r$ jumps forward and r jumps backward due to pulses of $\xi^{+}(s)$ and $\xi^{-}(s)$ are not considered in (3.17). For these paths a delta pulse of both $\xi^{+}(s)$ and $\xi^{-}(s)$ needs to occur. The probability of having any of these pulses in δ is $\lambda\delta$, so that the probability of both pulses is of order δ^2 and it does not contribute in (3.17). The physical difference between the GS and GA models becomes quite clear at this point. In the GS model, $\xi^{+}(s)=\xi^{-}(s)=\xi(s)$

and paths from N to $N+n$ including jumps backward contribute in (3.17). The total number of jumps $n+2r$ is a Poissonian variable of average $2\alpha\bar{\omega}$. Taking into account the number of differents paths, given by $\binom{n+2r}{r}$, we obtain (3.16). Basically, the GS model corresponds to a random walk in a stochastic time (the time scale μ becomes stochastic), while in the GA model we have two independent stochastic time scales for jumps forward and backward. As a consequence, for each realization of $\xi(s)$, the GS model respects the symmetry between jumps forward and backward of the random walk in the nondisordered case. Then the stochastic diffusion coefficient $1/2\, d\langle N^2\rangle/dt$ is given by $[\mu+\alpha\xi(t)]$, whose average is just μ, so that no modification occurs in the diffusion coefficient. On the other hand, in the GA model and for a realization of $\xi^+(s)$ and $\xi^-(s)$, a net drift appears. The average over realizations restores the symmetry in the effective process, but the underlying asymmetry speeds up the diffusion process. This explains the change found in the diffusion coefficient for the GA model.

3.2. Local Disorder

The EME for the dynamic RT model is obtained by taking the average of (2.17). To do this, it is important to recall the independence of $\xi_N(t)$ at different sites N. Noting that

$$(E^+ + E^- - 2)\,\alpha\xi_N(t)\,P(N,t)$$
$$= \alpha(\xi_{N+1}(t)\,E^+P(N,t) + \xi_{N-1}(t)\,E^-P(N,t) - 2\xi_N(t)\,P(N,t))$$

and making again use of (2.8), we have

$$\overline{\xi_N(t)\,P(N,t)} = \lambda\sum_{n=2}^{\infty}\frac{\{\omega^n\}}{n!}(-2\alpha)^{n-1}\,\overline{P(N,t)} = -\frac{\lambda}{2\alpha}\{e^{-2\alpha\omega}-1+2\alpha\omega\} \tag{3.19}$$

so that

$$\partial_t\overline{P(N,t)} = \left(\mu - \frac{\lambda}{2}\{e^{-2\alpha\omega}-1+2\alpha\omega\}\right)[E^+ + E^- - 2]\,\overline{P(N,t)} \tag{3.20}$$

which for exponentially distributed pulses becomes

$$\partial_t\overline{P(N,t)} = \left(\mu - \frac{2\lambda\alpha^2\bar{\omega}^2}{1+2\alpha\omega}\right)[E^+ + E^- - 2]\,\overline{P(N,t)} \tag{3.21}$$

The EMEs (3.20) and (3.21) are one-step MEs. Therefore they have the same mathematical properties as the original ME, (2.1), for ordinary random walk except for the modification of the jump rate μ. In particular, the solution of (3.20)–(3.21) is

$$P(N, t) = e^{-2\mu_{\text{eff}} t} I_N(2\mu_{\text{eff}} t) \tag{3.22}$$

where I_N is a modified Bessel function and the effective jump rate μ_{eff} is, for fixed and exponentially distributed pulses, respectively,

$$\mu_{\text{eff}}^{\text{F}} = \mu - \frac{\lambda}{2} (e^{-2\alpha\bar{\omega}} - 1 + \alpha\bar{\omega}) \tag{3.23}$$

$$\mu_{\text{eff}}^{\text{E}} = \mu - \frac{2\lambda\alpha^2\bar{\omega}^2}{1 + 2\alpha\bar{\omega}} \tag{3.24}$$

The explicit forms of the second and fourth moments are given in Table I. Our result indicates that in the dynamic RT model the introduction of disorder only amounts to a change in the time scale of evolution. Since $\mu > \mu_{\text{eff}}$, the process slows down. Note that the positivity of μ_{eff} is guaranteed by the requirement $\mu > \lambda\alpha\bar{\omega}$ and that the diffusion is slower for exponentially distributed pulses ($\mu_{\text{eff}}^{\text{F}} > \mu_{\text{eff}}^{\text{E}}$).

The main obvious difference between (3.20)–(3.21) and our results for global disorder is that we do not find now allowed transitions with jumps beyond the nearest neighbor sites. The reason for that can be understood by recalling the argument following (3.17): The accumulation of pulses per unit time was possible for global disorder because the same noise acts at each site during the path $N \to N+n$. We now have uncorrelated noise sources at each site acting locally and no accumulation of pulses can occur, by the same argument as the one given to see that paths involving jumps forward and backward could not contribute to $\bar{W}(N \to N+n)$ in the GA model: a path involving n pulses does not contribute in a probability per unit time. This argument is quite general for any model including local disorder given by independent noise sources at each site. In fact, we find below that for all the models with local disorder introduced in Section 2, the associated EME only includes one-step jumps.

The EME for the dynamic RB model (2.20) is obtained in a similar way. We first rewrite (2.20) as

$$\partial_t P(N, t) = \mu[E^+ + E^- - 2]\, P(N, t) + \alpha[E^+ - 1]\, \xi_N(t)\, P(N, t) + \alpha[E^- - 1]\, \xi_{N+1}(t)\, P(N, t) \tag{3.25}$$

The averages $\overline{\xi_N(t)\,P(N,t)}$ and $\overline{\xi_{N+1}(t)\,P(N,t)}$ are calculated using (2.8). The functional derivatives involved are

$$\frac{\delta^n P(N,t)}{\delta\xi_N(t)^n}=(-2\alpha)^{n-1}\,\alpha[E^- -1]\,P(N,t) \tag{3.26}$$

and

$$\frac{\delta^n P(N,t)}{\delta\xi_{N+1}(t)^n}=(-2\alpha)^{n-1}\,\alpha[E^+ -1]\,P(N,t) \tag{3.27}$$

so that

$$\overline{\xi_N(t)\,P(N,t)}=\frac{\alpha\lambda}{(-2\alpha)^2}\{e^{-2\alpha\omega}-1+2\alpha\omega\}[E^- -1]\,\overline{P(N,t)} \tag{3.28}$$

$$\overline{\xi_{N+1}(t)\,P(N,t)}=\frac{\alpha\lambda}{(-2\alpha)^2}\{e^{-2\alpha\omega}-1+2\alpha\omega\}[E^+ -1]\,\overline{P(N,t)} \tag{3.29}$$

Whit (3.28) and (3.29) substituted in (3.25), the EME turns out to be identical with (3.20) found for the RT model, as is the case for static disorder.[14]

We finally consider the EME for the model (2.22) with local asymmetric dynamic disorder. The average of (2.22) is more easily taken by rewriting it as

$$\begin{aligned}\partial_t P(N,t)=\mu[E^+ +E^- -2]\,P(N,t)+\alpha\xi^-_{N+1}(t)\,P(N+1,t)\\ +\alpha\xi^+_{N-1}(t)\,P(N-1,t)-\alpha[\xi^+_N(t)+\xi^-_N(t)]\,P(N,t)\end{aligned} \tag{3.30}$$

The functional derivatives involved in calculating $\overline{\xi^\pm_N(t)\,P(N,t)}$ from (2.8) are easily evaluated from (3.30). We obtain

$$\overline{\xi^\pm_N(t)\,P(N,t)}=-\frac{\lambda}{\alpha}\{e^{-\alpha\omega}-1+\alpha\omega\}\,\overline{P(N,t)} \tag{3.31}$$

Substituting (3.31) in the average of (2.22), we arrive at

$$\partial_t\overline{P(N,t)}=(\mu-\lambda\{e^{-\alpha\omega}-1+\alpha\omega\})[E^+ +E^- -2]\,\overline{P(N,t)} \tag{3.32}$$

This is again a one-step ME very similar to (3.20). The effective jump rate μ_{eff} is now

$$\mu^{\text{F}}_{\text{eff}}=\mu-\lambda(e^{-\alpha\bar\omega}-1+\alpha\bar\omega) \tag{3.33}$$

$$\mu^{\text{E}}_{\text{eff}}=\mu-\frac{\lambda\alpha^2\bar\omega^2}{1+\alpha\bar\omega} \tag{3.34}$$

Therefore we find an overall behavior very similar to the RB and RT models but with a larger jump rate μ_{eff}. Explicit results for the moments are given in Table I.

A general conclusion is that for models with local disorder we find a slower diffusion. This happens because the effect of disorder is to increase the average time spent by the random walker at a given site. This can be seen by considering the probability of having no jumps from a site N in a time interval δ. For example, for the RT model this probability is given by

$$\overline{\exp\left[-2\int_t^{t+\delta} ds(\mu+\alpha\xi_N(s))\right]}$$

3.3. Continuous Limit

The EMEs derived in Section 3.2 give a complete well-defined description of the different models introduced for random walk in dynamically disordered chains. In many cases a simpler continuous limit description would be desirable in the same way that the diffusion process (2.4) accounts for the main features of the discrete random walk defined by (2.1). The EMEs are a particularly useful starting point for obtaining meaningful continuous limits for models of random walk in disordered media. Our strategy is to consider the noise sources to be given and independent of the random walk dynamics. Therefore we keep the parameters $\bar{\omega}$ and λ fixed when taking the continuous limit. The parameter μ is scaled in the same way as for random walk in a nondisordered chain, $\mu l^2 = D$. The scaling with l of the coupling parameter α is determined by requiring that a finite effect of the disorder is found in the continuous limit. The continuous limit is taken as we did in (2.3), defining

$$\overline{P(x,t)} = \lim_{l\to 0} (1/l)\, \overline{P(N=x/l, t)}$$

For the GS model, Eqs. (3.2) and (3.3), we take $\alpha l^2 = A$. Introducing the notation $W = A\bar{\omega}$, it is straightforward to obtain from (3.2) for the case of fixed amplitude of the pulses

$$\partial_t \overline{P(x,t)} = (D-\lambda W)\,\partial_x^2 \overline{P(x,t)} + \lambda[\exp(W\partial_x^2)-1]\,\overline{P(x,t)} \tag{3.35}$$

For exponentially distributed pulses we obtain from (3.3)

$$\partial_t \overline{P(x,t)} = (D-\lambda W)\,\partial_x^2 \overline{P(x,t)} + \lambda \frac{W\partial_x^2}{1-W\partial_x^2}\,\overline{P(x,t)} \tag{3.36}$$

Equations (3.35) and (3.36) are not diffusion equations. They describe, at a continuous level, the modification of the diffusion process (2.4) due to dynamic disorder. Both equations contain even derivatives with respect to x to all orders. This implies the existence of noncontinuous sample paths. The positivity of the solutions of (3.35) and (3.36) is guaranteed by the condition imposed on the transition rates of the SME, which for the parameters appearing in the continuous limit becomes $D \geqslant \lambda W$. In fact, (3.36) has the same formal structure as the QCA (2.11) for which positivity of the solution has been explicitly proved.[9] One can also check that no divergent modes appear when (3.35) and (3.36) are Fourier transformed.

For the GA model (3.9)–(3.10) a different scaling of α is necessary. We now take $\alpha l = A$ and $W = A\bar{\omega}$, obtaining from (3.9) for fixed pulse amplitude

$$\partial_t \overline{P(x, t)} = D\, \partial_x^2 \overline{P(x, t)} + \lambda(e^{W\partial_x} + e^{-W\partial_x} - 2)\, \overline{P(x, t)} \tag{3.37}$$

and from (3.10) for exponentially distributed pulses

$$\partial_t \overline{P(x, t)} = D\, \partial_x^2 \overline{P(x, t)} + 2\lambda W^2 \frac{\partial_x^2}{1 - W^2 \partial_x^2}\, \overline{P(x, t)} \tag{3.38}$$

The scaling $\alpha l^2 = A$ used in the GS model gives here divergent coefficients in the equation for $\overline{P(x, t)}$. The difficulty is similar to the well-known one found when taking the continuous limit of a random walk in a nondisordered chain with unequal rates for jumps forward and backward.[15] We have already noticed that for each realization of $\xi^+(t)$, $\xi^-(t)$ we have in the GA model a nonvanishing drift. This is the reason behind the need of using a scaling $\alpha l = A$. The general comments made for (3.35) and (3.36) apply also to (3.37)–(3.38).

The difference found from the EMEs between the diffusion coefficient for the GS and GA models remains in the continuous limit. However, the basic difference between the two models in the continuous limit is seen upon comparing (3.35) with (3.37). Equation (3.37) implies the existence of jumps of the random walker of fixed amplitude W, while in (3.35) the jumps have no fixed amplitude. The reason behind this difference is that, as explained before, for the GA model only forward jumps contribute to $\bar{W}(N \to N \pm n)$ in (3.15). The number of jumps which contribute to $\bar{W}$ is a Poissonian variable with average $\alpha\bar{\omega}$. In the continuous limit, $\bar{W}(N \to N \pm n) \to \lambda\delta(x/l - \alpha\bar{\omega})$. In the GS model, paths from N to $N + n$ contain jumps forward and backward. The number of jumps is also a Poissonian variable of average $2\alpha\bar{\omega}$, but now the sign of individual jumps is random and independent from one jump to another. In the continuous limit the number of jumps is concentrated at its average value $2\alpha\bar{\omega} = 2W/l^2$.

Applying the central limit theorem, one finds that the displacement tends to a Gaussian distribution of zero average and variance $2W$. We will see below that in this way we recover the second term of (3.35), which corresponds to pulses of the noise.

For our three models with local disorder, the scaling $\alpha l^2 = A$ is used. The same continuous limit equation is found for the RT and RB [Eq. (3.20)] and local asymmetric [Eq. (3.32)] models ($W = A\bar{\omega}$):

$$\partial_t \overline{P(x, t)} = (D - \lambda W)\, \partial_x^2 \overline{P(x, t)} \tag{3.39}$$

Equation (3.39) is the same for fixed or exponentially distributed pulses. This result indicates that the three models coincide in the continuous limit with a diffusion process with effective diffusion coefficient $\bar{D} = D - \lambda W = \lim_{l \to 0} l^2 \mu_{\text{eff}}$. Diffusion becomes smaller due to disorder effects. Differences found in the discrete case are washed out in the continuous limit because only the systematic part of the noise $(-\alpha\lambda\bar{\omega})$ survives in this limit. In the continuous limit the contribution of the pulses of noise could only have an effect if its rate λ would scale as l^{-2}, while we are considering here λ as a constant in this limit. The continuous limit of the moments in Table I is easily obtained and can be also directly calculated from (3.35)–(3.39).

An interesting question is the representation in terms of sample paths of the processes found in the continuous limit. This question is answered by finding the SDE for the processes $x(t)$ equivalent to (3.35)–(3.39). For the models with local disorder this is an easy task, since (3.39) is equivalent to the usual Langevin equation for Brownian motion.

$$\dot{x}(t) = \chi(t) \tag{3.40}$$

where $\chi(t)$ is a Gaussian white noise of zero mean and correlation

$$\langle \chi(t)\, \chi(t') \rangle = 2(D - \lambda W)\, \delta(t - t') \tag{3.41}$$

We now consider the equations for the paths $x(t)$ for the models with global disorder. Equations (3.35)–(3.38) imply that $x(t)$ is not a diffusion process. The second term in the right-hand side of these equations is associated with jump processes. Quite generally, the Langevin-like equations for $x(t)$ should then contain a noise term associated with diffusion and other noise terms giving rise to the jump contributions. The explicit equation can be obtained by inspection of (3.35)–(3.38) and comparison with previous results. The GA (3.37)–(3.38) is easier to discuss: The comparison of (3.37) with (2.3) permits us to interpret (3.37) as a superposition of normal diffusion with coefficient D and random walk with jump rate λ in a lattice with spacing W. Likewise, the comparison of (3.38) with (2.11) identifies (3.38) as a superposition of normal diffusion with coefficient D and random walk with jump rate λ and jump amplitude exponentially distributed with mean value W. In summary, the appropriate SDE for $x(t)$ is

$$\dot{x}(t) = \chi(t) + \eta^{+}(t) - \eta^{-}(t) \tag{3.42}$$

where $\chi(t)$ and $\eta^{\pm}(t)$ are independent stochastic processes. Now, $\chi(t)$ is a Gaussian noise of zero mean and correlation $\langle \chi(t)\, \chi(t') \rangle = 2D\delta(t-t')$, and $\eta^{\pm}(t)$ are independent white Poisson noise (2.5b) with the same parameter λ. To obtain (3.37), $\eta^{\pm}(t)$ have pulses of fixed amplitude W, and to obtain (3.38), they have pulses with exponentially distributed amplitude of mean value W.

Equation (3.36) for the GS model is similar to (3.38) and for the same reason the stochastic equation for the paths $x(t)$ is also (3.42), but with different parameters of the noise sources $\chi(t)$ and $\eta^{\pm}(t)$. The parameter D is now replaced by $D - \lambda W$, $\eta^{\pm}(t)$ have parameter $\lambda/2$ instead of λ, and W is replaced by $W^{1/2}$.

The equation for $x(t)$ equivalent to (3.35) comes out to be

$$\dot{x}(t) = \chi(t) + \eta(t) \tag{3.43}$$

where $\chi(t)$ is again Gaussian white noise with correlation $\langle \chi(t)\, \chi(t') \rangle = 2(D - \lambda W)\, \delta(t-t')$ and $\eta(t)$ is a Poisson white noise (2.5b) with parameter λ and pulses of amplitude given by a Gaussian distribution:

$$\rho(\omega) = (4\pi W)^{-1/2} \exp(-\omega^2/4W) \tag{3.44}$$

The origin of these pulses with Gaussianly distributed amplitude was discussed above. To prove the equivalence of (3.43) and (3.35), one defines $P(x, t) \equiv \overline{\delta(x - x(t))}$ and averages the equation for $\delta(x - x(t))$:

$$\partial_t \delta(x - x(t)) = -\partial_x \chi(t)\, \delta(x - x(t)) - \partial_x \eta(t)\, \delta(x - x(t)) \tag{3.45}$$

The Gaussian average over $\chi(t)$ gives the diffusion term in (3.35). The average over $\eta(t)$ is taken using (2.8) and it reproduces the second term in the rhs of (3.35).

The result for GS with exponentialy distributed pulses is recovered by replacing the Gaussian (3.44) by the exponential resulting from averaging (3.44) with an exponential distribution for the pulse amplitude W.

4. CONTINUOUS LIMIT OF STOCHASTIC MASTER EQUATIONS

In this section we wish to address the following question: We have introduced models defined by SMEs whose average over stochastic disorder gives rise to EMEs. The continuous limits of the EMEs have been discussed. If one is only interested in the continuous limit, it might seem simpler to take the continuous limit of the SME and later make the average. If the continuous limit and stochastic averaging commute, the two procedures should give identical results. We will show that this not always the case and one might find that the continuous limit of the SMEs does not properly exist when the same continuous limit of the EMEs is well defined. Even when the two procedures commute, the continuous limit of the SME is often plagued with ill-defined quantities and great care is needed to obtain the correct result. Our conclusion and message is therefore that it is always safer and simpler to take the path through the EME in which no difficulties appear and everything is well defined at each stage of the calculation.

As a separate matter it is interesting to note that if the continuous limit of the random walk ME is taken before introducing sources of disorder, there is no natural way of defining some of the models considered here. For example, it is not simple to implement the idea behind the GA model if the starting point is the diffusion equation (2.4). In general, there are several models of random walk in a disordered chain which give different results in the continuous limit but which are difficult to introduce in the continuous limit of random walk in an ordered chain.

4.1. Global Disorder

For global disorder, noise sources are site independent and no difficulties are expected when taking the continuous limit of the SME. We give here the results for completeness. The continuous limit is defined as in Section 3.3 and the same scaling of parameters is used. For the GS model, $\alpha l^2 = A$, and the continuous limit of the SME (2.12) becomes

$$\partial_t P(x, t) = [D + A\xi(t)]\, \partial_x^2 P(x, t) \tag{4.1}$$

Likewise, for the GA model with $\alpha l = A$ we obtain from (2.13)

$$\partial_t P(x, t) = \{D\partial_x^2 + A[\xi^-(t) - \xi^+(t)]\}\, \partial_x]\, P(x, t) \tag{4.2}$$

At this level of description the differences between the GS and GA models are clear. Equation (4.1) describes a pure diffusion process in which the diffusion coefficient is a positive-definite, time-dependent stochastic quantity. The stochastic part averages to zero and it does not modify the coefficient $d \equiv \overline{\langle N^2 \rangle}/(2t)$. On the other hand, (4.2) includes a stochastic drift due to the independence of $\xi^-(t)$ and $\xi^+(t)$. This stochastic drift was already mentioned before. It averages to zero, but it contributes to the coefficient d, giving rise to faster diffusion.

The continuous limit (3.35)–(3.38) of the EMEs can be obtained from (4.1) and (4.2) by alternative methods. One is showing the equivalences of (4.1) and (4.2) with (3.43) and (3.42) respectively. More directly, the average of (4.1) and (4.2) can be taken using once more the general formula (2.8). In (4.1) we obtain

$$\overline{\xi(t)\,\partial_x^2 P(x,t)} = \lambda\bar{\omega}\,\partial_x^2 \sum_{k=1}^{\infty} (\bar{\omega}A\,\partial_x^2)^k\,\overline{P(x,t)} \tag{4.3}$$

while for (4.2) we have

$$\overline{\xi(t)\,\partial_x P(x,t)} = \mp\lambda\bar{\omega}\,\partial_x \sum_{k=1}^{\infty} (\bar{\omega}A\,\partial_x)^k\,\overline{P(x,t)} \tag{4.4}$$

Equations (4.3) and (4.4) are for noise with exponentially distributed amplitude of the pulses. Introducing them in the average of (4.1) and (4.2), we recover (3.36) and (3.38). A similar derivation can be given for noise with fixed amplitude.

4.2. Local Disorder

We first consider the RT and RB models, and the more delicate local model with intrinsic asymmetry is discussed later. To find the continuous limit of the SME (2.17) associated with the RT model, we rewrite (2.17) as

$$\partial_t P^l(x,t) = (e^{l\partial_x} + e^{-l\partial_x} - 2)[\mu + \alpha\xi^l(x,t)]\,P^l(x,t) \tag{4.5}$$

where we use the same notation as in (2.3). Taking the limit $l \to 0$ with the scaling $A = \alpha l^2$, we obtain

$$\partial_t P(x,t) = \partial_x^2[D + A\xi(x,t)]\,P(x,t) \tag{4.6}$$

Likewise for the SME (2.20) associated with the RB model we have

$$\begin{aligned}\partial_t P^l(x,t) = {}& \mu(e^{l\partial_x} + e^{-l\partial_x} - 2)\,P^l(x,t)\\ & - (e^{l\partial_x} - 1)\,\xi^l(x,t)(e^{-l\partial_x} - 1)\,P^l(x,t)\end{aligned} \tag{4.7}$$

which in the limit $l \to 0$ with $A = \alpha l^2$ reduces to

$$\partial_t P(x,t) = D\,\partial_x^2 P(x,t) + A\,\partial_x \xi(x,t)\,\partial_x P(x,t) \tag{4.8}$$

It is important to note here the different position of $\xi(x, t)$ relative to the operation ∂_x in (4.6) and (4.8). For static disorder (4.6) and (4.8) concide with the continuous limits used in ref. 16. Other continuous limits have been proposed in the literature.[4] The formal derivation of (4.6) and (4.8) can be justified by taking the limit $l \to 0$ in a weak sense, that is, considering only averaged quantities. However, the real problem is not with the form of (4.6), (4.8), but rather with the meaning to be given to $\xi(x, t) = \lim_{l \to 0} \xi^l(x, t)$. In fact, to give sense to averages which involve $\xi(x, t)$, one is forced to go back to the original discrete version, so that equations like (4.6) or (4.8) are of little operational significance. To begin with, we need a precise definition of $\xi^l(x, t)$ for a continuous variable x. We take $\xi^l(x, t)$ to coincide with $\xi_N(t)$ when $N - l/2 < x < N + l/2$:

$$\xi^l(x, t) \equiv \sum_N H(x - Nl)\, \xi_N(t) \tag{4.9}$$

$$H(x - Nl) \equiv \theta\left(x - Nl + \frac{l}{2}\right) - \theta\left(x - Nl - \frac{l}{2}\right) \tag{4.10}$$

where $\theta(x)$ is the Heaviside step function. Similarly, and being precise in the definition used in (2.3),

$$P^l(x, t) \equiv \frac{1}{l} \sum H(x - Nl)\, P(N, t) \tag{4.11}$$

The statistical properties of the stochastic process $\xi(x, t)$ are determined by the set of its cumulants. Given that for independent Poisson white noise

$$\begin{aligned} &\langle\langle \xi_{N_1}(t_1)\, \xi_{N_2}(t_2) \cdots \xi_{N_n}(t_n) \rangle\rangle \\ &\quad = \lambda\{\omega^n\}\, \delta(t_1 - t_2)\, \delta(t_1 - t_3) \cdots \delta(t_1 - t_n)\, \delta_{N_1 N_2} \delta_{N_1 N_3} \cdots \delta_{N_1 N_n} \end{aligned} \tag{4.12}$$

we have

$$\begin{aligned} &\langle\langle \xi^l(x_1, t_1) \cdots \xi^l(x_n, t_n) \rangle\rangle \\ &\quad = \lambda\{\omega^n\}\, \delta(t_1 - t_2) \cdots \delta(t_1 - t_n)\, H(x_1 - x_2) \cdots H(x_1 - x_n) \end{aligned} \tag{4.13}$$

so that

$$\begin{aligned} &\langle\langle \xi(x_1, t_1) \cdots \xi(x_n, t_n) \rangle\rangle \\ &\quad = \lambda\{\omega^n\}\, \delta(t_1 - t_2) \cdots \delta(t_1 - t_n)\, \delta_{x_1 x_2} \cdots \delta_{x_1 x_n} \end{aligned} \tag{4.14}$$

[4] For example, in ref. 7 and using the scaling $A = \alpha l$ the proposed equation for the RB model is

$$\partial_t P(x, t) = D\, \partial_x^2 P(x, t) + A[\xi(x^+, t) - \xi(x, t)]\, \partial_x P(x, t)$$

Averages calculated with such an equation give results with no effect of the disorder. This is not immediate to see when dealing with $\xi(x, t)$, but it is obvious if one takes the continuous limit of the EME (3.20) with the same scaling $A = \alpha l$, which gives pure diffusion.

where δ_{x_i, x_j} is a Kronecker delta. The difficulties which appear in deriving a continuous limit without taking first the average of the SME are due to the handling of these Kronecker deltas for a continuous variable x.

To make such difficulties clear, let us attempt the evaluation of $\overline{\xi(x, t)\, P(x, t)}$ in (4.6). To do this, we need the generalization of (2.8):

$$\overline{\xi(x, t)\, P(x, t)} = \lambda \sum_{n=2}^{\infty} \frac{\{\omega^n\}}{n!} \int dx_1 \cdots dx_{n-1}\, dt_1 \cdots dt_{n-1} \times \langle\!\langle \xi(x, t)\, \xi(x_1, t_1) \cdots \xi(x_{n-1}, t_{n-1}) \rangle\!\rangle \times \frac{\overline{\delta^{n-1} P(x, t)}}{\delta\xi(x_1, t_1) \cdots \delta\xi(x_{n-1}, t_{n-1})} \tag{4.15}$$

When substituting (4.14) in (4.15), one immediately finds quantities requiring a careful interpretation.[5] A way to avoid these problems which permits us to obtain the correct average of (4.6) is to define

$$\overline{\xi(x, t)\, P(x, t)} = \lim_{l \to 0} \overline{\xi^l(x, t)\, P^l(x, t)} \tag{4.16}$$

The rhs of (4.16) is evaluated using (4.15) for ξ^l and P^l and taking into account (4.13):

$$\overline{\xi^l(x, t)\, P^l(x, t)} = \lambda \sum_{n=2}^{\infty} \frac{\{\omega^n\}}{n!} \int_{x-l/2}^{x+l/2} dx_1 \cdots \int_{x-l/2}^{x+l/2} dx_{n-1} \frac{\overline{\delta^{n-1} P(x, t)}}{\delta\xi(x_1, t) \cdots \delta\xi(x_{n-1}, t)} \tag{4.17}$$

The functional derivatives in (4.17) are evaluated, for the RT model, from (4.5):

$$\frac{\delta P^l(x, t)}{\delta\xi^l(x', t)} = \alpha(\delta(x + l - x')\, P^l(x + l, t) + \delta(x - l - x')\, P^l(x - l, t) - 2\delta(x - x')\, P^l(x, t) \tag{4.18}$$

[5] For example, one has, for $n = 2$,

$$\int dx_1\, \delta_{x, x_1} A\, \partial_x^2\, \delta(x - x_1)\, \bar{P}(x, t)$$

The correct interpretation of this expression is found to be

$$\lim_{l \to 0} (-2A/l^2)\, \bar{P}(x, t)$$

so that if $x' \in (x - l/2, x + l/2)$,

$$\frac{\delta P^l(x, t)}{\delta \xi^l(x', t)} = -2\alpha\delta(x - x')\, P^l(x, t) \tag{4.19}$$

and

$$\overline{\xi^l(x, t)\, P^l(x, t)}$$
$$= \lambda \sum_{n=2}^{\infty} \frac{\{\omega^n\}}{n!} (-2\alpha)^{n-1}\, \overline{P^l(x, t)}$$
$$= \left(-\frac{\lambda}{2\alpha} \{e^{-2\alpha\omega} - 1\} - \lambda\bar{\omega} \right) \overline{P^l(x, t)} \xrightarrow[A = \alpha l^2]{l \to 0} -\lambda\bar{\omega}\, \overline{P(x, t)} \tag{4.20}$$

An important point to note in (4.20) is that the finite result in the limit $l \to 0$ is the sum of an alternating series in which any individual term diverges. The continuous-limit equation (3.39) for the RT model is reobtained by taking the average of (4.6) with (4.16) and (4.20). The same result (3.39) for the RB model is obtained from (4.8) using (4.16)–(4.17) and calculating the functional derivatives from (4.7).

The path followed here to obtain the continuous limit for the probability distribution averaged over the sources of disorder is even more subtle for the local model with intrinsic asymmetry (2.22). The SME (2.22) can be rewritten as

$$\partial_t P^l(x, t) = \mu(e^{l\partial_x} + e^{-l\partial_x} - 2)\, P^l(x, t)$$
$$+ \alpha[(e^{l\partial_x} - 1)\, \xi^l(x, t) + (e^{-l\partial_x} - 1)\, \xi^l(x, t)]\, P^l(x, t) \tag{4.21}$$

Introducing the same scaling $\alpha l^2 = A$ that was needed to obtain the continuous limit of the EME (3.32), we have

$$\partial_t P(x, t) = D\partial_x^2 P(x, t) + \bar{A}\partial_x[\xi^-(x, t) - \xi^+(x, t)]\, P(x, t)$$
$$+ \tfrac{1}{2}A\, \partial_x^2[\xi^-(x, t) + \xi^+(x, t)]\, P(x, t) \tag{4.22}$$

where $\bar{A} = \lim_{l \to 0}(A/l)$. This indicates that, properly speaking, the limit under consideration does not exist, because the stochastic drift term in (4.22) diverges. However, the equation (3.39) for $\overline{P(x, t)}$ in the continuous limit can be obtained by taking the average of (4.22). The reason is that, following the same procedure as above, it is found that

$$\overline{\xi^-(x, t)\, P(x, t)} = \overline{\xi^+(x, t)\, P(x, t)} = -\lambda\bar{\omega}\overline{P(x, t)} \tag{4.23}$$

This means that when taking the average of (4.22) there is a cancellation of two divergent terms. In summary, for the model (2.22) the average over the noise realizations and the continuous limit are operations that do not commute in the sense that (4.22) contains divergent quantities. This example makes clear the advantages of the method followed in Section 3, that is, first make the average and then take the continuous limit.

The main question addressed in this section is the correct interpretation of $\xi(x, t)$ and its statistical properties. One is tempted to think that correlations like $\langle\xi(x, t)\,\xi(x', t')\rangle$ have to be proportional to $\delta(x-x')$. Even if this were so, stochastic partial differential equations like (4.6), (4.8), or (4.22) require some prescription to be interpreted, which is usually given in terms of a lattice system.[17] In our case, the analysis of the discrete model leads to correlations proportional to $\delta_{x,x'}$. We recall, however, that we have defined the continuous limit of our models keeping the noise parameters $\bar{\omega}$ and λ fixed. If we introduce a scaling $\Lambda = \lambda l^{-1}$, $\Omega = \omega l$, (4.14) becomes

$$\langle\!\langle \xi(x_1, t_1)\cdots\xi(x_n, t_n)\rangle\!\rangle$$
$$= \Lambda\{\Omega^n\}\,\delta(t_1-t_2)\cdots\delta(t_1-t_n)\,\delta(x_1-x_2)\cdots\delta(x_1-x_n) \quad (4.24)$$

The average of equations like (4.6) and (4.8) with $\xi(x, t)$ given by (4.24) leads to the occurrence of many divergent quantities. In particular, occurrence of $\delta(0)$ can be seen in ref. 7. With this scaling of the noise parameters, $\Lambda = \lambda l^{-1}$ and $\Omega = \omega l$, and keeping the coupling constant α fixed taking $l \to 0$, the continuous limit of the EME is just the diffusion equation (2.4), so that no effect of disorder remains in this limit. Since no divergence is found in this last procedure, we see another example of the noncommutation of the continuous limit and the averaging over disorder sources. We finally note that if the limit $l \to 0$ is taken, scaling the noise parameters as $\Lambda = \lambda l^{-1}$, $\bar{\Omega} = \bar{\omega} l$, but also the coupling parameter $\alpha = A l^{-2}$ as we have generally done, the EME for any of the models with local disorder has a well-defined continuous limit:

$$\partial_t \overline{P(x, t)} = (D - A\Lambda\bar{\Omega})\,\partial_x^2 \overline{P(x, t)} \quad (4.25)$$

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